

# From full rank subdivision schemes to multichannel wavelets: A constructive approach

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## Abstract

In this paper, we describe some recent results obtained in the context of vector subdivision schemes which possess the so-called full rank property. Such kind of schemes, in particular those which have an interpolatory nature, are connected to matrix refinable functions generating orthogonal multiresolution analyses for the space of vector-valued signals. Corresponding multichannel (matrix) wavelets can be defined and their construction in terms of a very efficient scheme is given. Some examples illustrate the nature of these matrix scaling functions/wavelets.

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## 1. Introduction

This paper deals with the construction of the proper "wavelet" tools for the analysis of functions which are *vector-valued*. Such type of functions arises naturally in many applications where the data to be processed are samples of vector-valued functions, even in several variables (e.g. electroencephalographic measurements, seismic waves, color images).

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Up to now, most signal processing methods handle vector valued data component by component and thus ignore the possible relationship between several of the components. In reality, on the other hand, such measurements are usually related and thus should be processed as “complete” vectors. While it is obvious that for vector valued signals a standard scalar wavelet analysis does not take into account the correlations among components, even an approach based on the well known *multiwavelets*, introduced in [7], [12] and widely studied (see [8] and the exhaustive references therein), is only apparently justified by the “vectorization” step required in order to apply them. Indeed, multiwavelets still generate multiresolution analyses for the space of scalar-valued functions.

In [13], [14] *matrix wavelets* have been proposed for the analysis of matrix valued signals while in [1] the concept of *multichannel wavelet* (MCW) and *multichannel multiresolution analysis* (MCMRA) have been introduced for processing vector-valued data. Actually, the two concepts look very similar: the crucial point of both is the existence of a matrix refinable function, which satisfies the so-called *full rank condition*. However, on one hand, the matrix analysis scheme proposed in [13], [14] coincides with the applications of a fixed number of MCW schemes, on the other hand, only a very special characterization of filters is proposed, which gives rise to interpolatory orthonormal matrix refinable functions. Thanks to this special structure, the corresponding wavelet filters are easily found as in the scalar situation. Nevertheless, in [13], [14] the authors do not provide either any constructive strategy nor any example. Some examples of  $2 \times 2$  matrix wavelets are derived in [15], but they strongly rely on the sufficient condition for convergence described in [13], and belong again to a very special class. In passing, we also mention reference [6] since, in spite of the title, it actually contains examples of wavelet filters specifically designed for vector-fields.

In some recent papers [2], [3], [4] we investigated *full rank interpolatory schemes* and showed their connection to matrix refinable functions and multichannel wavelets. In particular we proved that there is a connection between *cardinal* and *orthogonal* matrix refinable functions which is based on the *spectral factorization* of the (positive definite) symbol related to the cardinal function, thus, as a by-product, giving a concrete way to obtain orthogonal matrix scaling functions. The corresponding multichannel wavelets, whose existence is proved in [1], can be found in terms of a symbol which satisfies *matrix quadrature mirror filter* equations.

In this paper, we show how the solution of matrix quadrature mirror filter equations can be found. In particular we give an efficient and constructive scheme which makes use again of spectral factorization techniques and of a matrix completion algorithm based on the resolution of generalized Bezout identities.

The paper is organized as follows. In Section 2 all the needed notation and definitions are set. Some introductory results on full rank vector subdivision schemes are also presented and their connection to orthogonal full rank refinable functions is shown. In Section 3, the concepts of MCMRA and MCW are revised

while their orthogonal counterparts are considered next in Section 4. Section 5 deals with the construction of multichannel wavelets and an explicit algorithm for such construction is presented. In Section 6, starting from some interpolatory full rank positive definite symbols, the associated multichannel wavelets with two and three channels are constructed and shown. Some conclusions are derived in Section 7.

## 2. Notation and basic facts on vector subdivision schemes

For  $r, s \in \mathbb{N}$  we write a matrix  $\mathbf{A} \in \mathbb{R}^{r \times s}$  as  $\mathbf{A} = (a_{jk} : j = 1, \dots, r, k = 1, \dots, s)$  and denote by  $\ell^{r \times s}(\mathbb{Z})$  the space of all  $r \times s$ -matrix valued bi-infinite sequences,  $\mathcal{A} = (\mathbf{A}_j \in \mathbb{R}^{r \times s} : j \in \mathbb{Z})$ . For notational simplicity we write  $\ell^r(\mathbb{Z})$  for  $\ell^{r \times 1}(\mathbb{Z})$  and  $\ell(\mathbb{Z})$  for  $\ell^1(\mathbb{Z})$  and denote vector sequences by lowercase letters.

By  $\ell_2^{r \times s}(\mathbb{Z})$  we denote the space of all sequences which have finite 2 norm defined as

$$\|\mathcal{A}\|_2 := \left( \sum_{j \in \mathbb{Z}} |\mathbf{A}_j|_2^2 \right)^{1/2}, \quad (1)$$

where  $|\cdot|_2$  denotes the 2 operator norm for  $r \times s$  matrices.

Moreover,  $L_2^{r \times s}(\mathbb{R})$  will denote the Banach space of all  $r \times s$ -matrix valued functions on  $\mathbb{R}$  with components in  $L_2(\mathbb{R})$  and norm

$$\|\mathbf{F}\|_2 = \left( \sum_{k=1}^s \sum_{j=1}^r \int_{\mathbb{R}} |F_{jk}(x)|^2 dx \right)^{1/2}. \quad (2)$$

For a matrix function and a matrix sequence of suitable sizes we introduce the *convolution* “ $*$ ” defined as,

$$(\mathbf{F} * \mathcal{B}) := \sum_{k \in \mathbb{Z}} \mathbf{F}(\cdot - k) \mathbf{B}_k.$$

For two matrix functions  $\mathbf{F}, \mathbf{G} \in L_2^{r \times s}(\mathbb{R})$ , their inner product is defined as

$$\langle \mathbf{G}, \mathbf{F} \rangle = \int_{\mathbb{R}} \mathbf{G}^H(x) \mathbf{F}(x) dx.$$

and we have

$$\|\mathbf{F}\|_2 = (\text{trace} \langle \mathbf{F}, \mathbf{F} \rangle)^{\frac{1}{2}}.$$

Let now  $\mathcal{A}$  be a *matrix sequence*  $\mathcal{A} = (\mathbf{A}_j \in \mathbb{R}^{r \times r} : j \in \mathbb{Z}) \in \ell^{r \times r}(\mathbb{Z})$ . Let  $c$  be any bi-infinite *vector sequence*  $c = (c_j \in \mathbb{R}^r : j \in \mathbb{Z})$ . The *vector subdivision operator*  $S_{\mathcal{A}}$ , based on the matrix mask  $\mathcal{A}$ , acts on  $c$  by means of

$$S_{\mathcal{A}}c = \left( \sum_{k \in \mathbb{Z}} \mathbf{A}_{j-2k} c_k : j \in \mathbb{Z} \right).$$

A *vector subdivision scheme* consists of iterative applications of a vector subdivision operator starting from an initial vector sequence  $c \in \ell^r(\mathbb{Z})$ , namely:

$$\begin{aligned} c^0 &:= c \\ c^n &:= S_{\mathcal{A}} c^{n-1} = S_{\mathcal{A}}^n c^0 \quad n \geq 1. \end{aligned}$$

It is  *$L_2$ -convergent* if there exists a *matrix-valued function*  $\mathbf{F} \in L_2^{r \times r}(\mathbb{R})$  such that

$$\lim_{n \rightarrow \infty} 2^{-n/2} \|\mu^n(\mathbf{F}) - S_{\mathcal{A}}^n \delta \mathbf{I}\|_2 = 0, \quad (3)$$

where the mean value operator at level  $n \in \mathbb{N}$  is defined as

$$(\mu^n(\mathbf{F}))_j := 2^n \int_{2^{-n}(j+[0,1])} \mathbf{F}(t) dt, \quad j \in \mathbb{Z}.$$

The matrix-valued function  $\mathbf{F}$  associated in such a way with a convergent subdivision scheme is called the *basic limit function* and it is *refinable* with respect to  $\mathcal{A}$ , that is

$$\mathbf{F} = \mathbf{F} * \mathbf{A}(2 \cdot) = \sum_{j \in \mathbb{Z}} \mathbf{F}(2 \cdot - j) \mathbf{A}_j.$$

The *symbol* of a subdivision scheme  $S_{\mathcal{A}}$  or of the mask  $\mathcal{A}$  is defined as:

$$\mathbf{A}(z) = \sum_{j \in \mathbb{Z}} \mathbf{A}_j z^j, \quad z \in \mathbb{C} \setminus \{0\},$$

and the *subsymbols*,  $\mathbf{A}_{\varepsilon}(z) = \sum_{j \in \mathbb{Z}} \mathbf{A}_{\varepsilon+2j} z^j$ ,  $z \in \mathbb{C} \setminus \{0\}$ ,  $\varepsilon \in \{0, 1\}$ , are related to the symbol by

$$\mathbf{A}(z) = \mathbf{A}_0(z^2) + z \mathbf{A}_1(z^2), \quad z \in \mathbb{C} \setminus \{0\}. \quad (4)$$

Next, we define the two  $r \times r$  matrices

$$\mathbf{A}_{\varepsilon} := \mathbf{A}_{\varepsilon}(1) = \sum_{j \in \mathbb{Z}} \mathbf{A}_{\varepsilon+2j}, \quad \varepsilon \in \{0, 1\}$$

and their joint 1-eigenspace

$$\mathcal{E}_{\mathcal{A}} := \{\mathbf{y} \in \mathbb{R}^r : \mathbf{A}_0 \mathbf{y} = \mathbf{A}_1 \mathbf{y} = \mathbf{y}\}.$$

The *rank* of the mask  $\mathcal{A}$  or of the subdivision scheme  $S_{\mathcal{A}}$  is the number

$$R(\mathcal{A}) := \dim \mathcal{E}_{\mathcal{A}}$$

satisfying  $1 \leq R(\mathcal{A}) \leq r$  for convergent schemes. In particular,  $S_{\mathcal{A}}$  is said to be of *full rank* if  $R(\mathcal{A}) = r$ .

The following proposition gives equivalent characterizations of full rank schemes [2]

**Proposition 1.** *Let  $S_{\mathcal{A}}$  be a vector subdivision scheme with symbol  $\mathbf{A}(z)$ . Then the following statements are equivalent:*

- i)  $S_{\mathcal{A}}$  is of full rank;
- ii) the symbol satisfies  $\mathbf{A}(1) = 2\mathbf{I}$  and  $\mathbf{A}(-1) = \mathbf{0}$ , or, equivalently, there exists a matrix mask  $\mathbf{B} \in \ell^{r \times r}(\mathbb{Z})$  such that  $\mathbf{A}(z) = (z+1)\mathbf{B}(z)$ , with  $\mathbf{B}(1) = \mathbf{I}$  (scalar-like factorization);
- iii) the scheme preserves all vector constant data, i.e. whenever  $\mathbf{c}_j = \mathbf{d}$ ,  $j \in \mathbb{Z}$ , for some  $\mathbf{d} \in \mathbb{R}^r$ , then also  $(S_{\mathcal{A}}\mathbf{c})_j = \mathbf{d}$ ,  $j \in \mathbb{Z}$ ;
- iv) the basic limit function  $\mathbf{F}$  is a partition of the identity, i.e.  $\sum_{j \in \mathbb{Z}} \mathbf{F}(\cdot - j) = \mathbf{I}$ .

A full rank vector subdivision scheme is *interpolatory* if it is characterized by the property

$$(S_{\mathcal{A}}\mathbf{c})_{2j} = \mathbf{c}_j, \quad j \in \mathbb{Z},$$

or, equivalently,

$$\mathbf{A}_{2j} = \delta_{j0} \mathbf{I}, \quad j \in \mathbb{Z}.$$

Its associated basic limit function, whenever it exists, turns out to be *cardinal*, i.e.,

$$\mathbf{F}(j) = \delta_{j0} \mathbf{I}, \quad j \in \mathbb{Z}.$$

The connection between interpolatory and full rank subdivision schemes is expressed by the following result [2]:

**Proposition 2.** *An interpolatory subdivision scheme with symbol  $\mathbf{A}(z)$  is of full rank if and only if  $\mathbf{A}(1) = 2\mathbf{I}$ .*

### 3. Multichannel multiresolution analysis and multichannel wavelets

As in the scalar situation, a *multichannel multiresolution analysis* (MCMRA) in the space  $L_2^r(\mathbb{R})$  of square integrable vector valued functions can be defined by a nested sequence  $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$  of closed subspaces of  $L_2^r(\mathbb{R})$  with the properties that

1. they are *shift invariant*

$$\mathbf{h} \in V_j \iff \mathbf{h}(\cdot + k) \in V_j, \quad k \in \mathbb{Z};$$

2. they are *scaled* versions of each other

$$\mathbf{h} \in V_0 \iff \mathbf{h}(2^j \cdot) \in V_j;$$

3.  $\bigcap_{j \in \mathbb{Z}} V_j = \{[0, 0 \dots 0]^T\}$ ,  $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L_2^r(\mathbb{R})$ ;

4. the space  $V_0$  is generated by the integer translates of  $r$  function vectors, that is there exist  $\mathbf{f}^i = (f_1^i, \dots, f_r^i)^T \in L_2^r(\mathbb{R})$ ,  $i = 1, \dots, r$ , such that

$$i) \quad V_0 = \text{span} \{ \mathbf{f}^i(\cdot - k) : k \in \mathbb{Z}, i = 1, \dots, r \},$$

ii) there exist two constants  $K_1 \leq K_2$  such that

$$K_1 \left( \sum_{i=1}^r \sum_{k \in \mathbb{Z}} |c_k^i|^2 \right)^{1/2} \leq \left\| \sum_{i=1}^r \sum_{k \in \mathbb{Z}} c_k^i \mathbf{f}^i(\cdot - k) \right\|_2 \leq K_2 \left( \sum_{j=1}^r \sum_{k \in \mathbb{Z}} |c_k^j|^2 \right)^{1/2} \quad (5)$$

for any  $r$  scalar sequences  $c^i = \{c_k^i\} \in \ell_2(\mathbb{Z})$ ,  $i = 1, \dots, r$ .

Now, let  $\mathbf{h} = (h_1, \dots, h_r)^T \in V_0 \subset L_2^r(\mathbb{R})$ . Then there exist  $r$  scalar sequences  $c^i = \{c_k^i \in \mathbb{R}, : k \in \mathbb{Z}\} \in \ell_2(\mathbb{Z})$ ,  $i = 1, \dots, r$ , such that

$$\mathbf{h} = \sum_{k \in \mathbb{Z}} (c_k^1 \mathbf{f}^1(\cdot - k) + \dots + c_k^r \mathbf{f}^r(\cdot - k)),$$

that is

$$\begin{aligned} \begin{pmatrix} h_1 \\ \vdots \\ h_r \end{pmatrix} &= \sum_{k \in \mathbb{Z}} \left( c_k^1 \begin{pmatrix} f_1^1(\cdot - k) \\ \vdots \\ f_r^1(\cdot - k) \end{pmatrix} + \dots + c_k^r \begin{pmatrix} f_1^r(\cdot - k) \\ \vdots \\ f_r^r(\cdot - k) \end{pmatrix} \right) \\ &= \sum_{k \in \mathbb{Z}} \underbrace{\begin{pmatrix} f_1^1(\cdot - k) & \dots & f_1^r(\cdot - k) \\ \vdots & & \vdots \\ f_r^1(\cdot - k) & \dots & f_r^r(\cdot - k) \end{pmatrix}}_{\mathbf{F}(\cdot - k)} \underbrace{\begin{pmatrix} c_k^1 \\ \vdots \\ c_k^r \end{pmatrix}}_{\mathbf{c}_k}. \end{aligned}$$

Thus, any  $\mathbf{h} \in V_0$  can be written as  $\mathbf{h} = \mathbf{F} * \mathbf{c} = \sum_{k \in \mathbb{Z}} \mathbf{F}(\cdot - k) \mathbf{c}_k$ , where  $\mathbf{F} \in L_2^{r \times r}(\mathbb{R})$  and  $\mathbf{c} = (\mathbf{c}_k \in \mathbb{R}^r : k \in \mathbb{Z}) \in \ell_2^r(\mathbb{Z})$ . Since  $\mathbf{f}^1, \dots, \mathbf{f}^r \in V_0 \subset V_1$ , we have

$$\mathbf{f}^i = \sum_{k \in \mathbb{Z}} \mathbf{F}(2 \cdot - k) \mathbf{a}_k^i, \quad i = 1, \dots, r,$$

for some vector sequences  $\mathbf{a}^i = (\mathbf{a}_k^i \in \mathbb{R}^r, : k \in \mathbb{Z}) \in \ell_2^r(\mathbb{Z})$ ,  $i = 1, \dots, r$ . Thus, the matrix-valued function

$$\mathbf{F} = (\mathbf{f}^1 | \dots | \mathbf{f}^r) \in L_2^{r \times r}(\mathbb{R})$$

satisfies the *matrix refinement equation*

$$\mathbf{F} = \sum_{k \in \mathbb{Z}} \mathbf{F}(2 \cdot - k) \mathbf{A}_k \quad (6)$$

where  $\mathcal{A} = (\mathbf{A}_k = (\mathbf{a}_k^1 | \dots | \mathbf{a}_k^r) \in \mathbb{R}^{r \times r} : k \in \mathbb{Z})$  is the *refinement mask*.

Now, since the subdivision scheme  $S_{\mathcal{A}}$  applied to the initial sequence  $\delta \mathbf{I}$  converges to a matrix function with  $r$  stable columns it follows that  $\mathcal{E}_{\mathcal{A}} = \mathbb{R}^r$  so that

$$\mathbf{A}_{\varepsilon} = \sum_{j \in \mathbb{Z}} \mathbf{A}_{\varepsilon - 2j} = \mathbf{I}, \quad \varepsilon \in \{0, 1\},$$

i.e.  $\mathbf{A}(1) = 2\mathbf{I}$ ,  $\mathbf{A}(-1) = \mathbf{0}$ , which is the full rank property of the mask  $\mathcal{A}$ .

The "joint stability" condition (5) on the function vectors  $\mathbf{f}^j$ ,  $j = 1, \dots, r$ , implies that the function  $\mathbf{F}$  is stable in the sense that there exist two constants  $K_1 \leq K_2$  such that

$$K_1 \|c\|_2 \leq \|\mathbf{F} * c\|_2 \leq K_2 \|c\|_2, \quad c \in \ell_2^r(\mathbb{Z}).$$

Summarizing the described properties, the following important result holds true:

**Theorem 1.** *The subspaces  $V_j \subset L_2^r(\mathbb{R})$ ,  $j \in \mathbb{Z}$ , form an MCMRA if there exists a full rank stable matrix refinable function  $\mathbf{F} \in L_2^{r \times r}(\mathbb{R})$  such that*

$$V_j = \{\mathbf{F} * c(2^j \cdot) : c \in \ell_2^r(\mathbb{Z})\}, \quad j \in \mathbb{Z}.$$

In analogy to the scalar case, we call  $\mathbf{F}$  *matrix scaling function*.

#### 4. Orthogonal multichannel multiresolution analysis

For many application purposes, *orthogonal MCMRAs* are the most interesting. They are equivalently characterized by the following properties:

1. the space  $V_0$  is generated by orthonormal integer translates of the function vectors  $\mathbf{f}^j$ ,  $j = 1, \dots, r$ ;
2. the matrix scaling function  $\mathbf{F}$  is orthonormal that is

$$\langle \mathbf{F}, \mathbf{F}(\cdot + k) \rangle := \int_{\mathbb{R}} \mathbf{F}^H(x) \mathbf{F}(x + k) dx = \delta_{k0} \mathbf{I}, \quad k \in \mathbb{Z}.$$

The key for the construction of orthonormal matrix scaling functions (and the associated multichannel wavelets) is their connection with interpolatory vector subdivision schemes, as established in the following Theorem where the *canonical spectral factor*  $\mathbf{A}(z)$  of a positive definite interpolatory full rank symbol  $\mathbf{C}(z)$  is the symbol satisfying  $\mathbf{C}(z) = \frac{1}{2} \mathbf{A}^H(z) \mathbf{A}(z)$  and  $\mathbf{A}(1) = 2\mathbf{I}$ ,  $\mathbf{A}(-1) = \mathbf{0}$ .

**Theorem 2.** [2] *Let  $\mathbf{C}(z)$  be the symbol of an interpolatory full rank vector subdivision scheme. Let  $\mathbf{C}(z)$  be symmetric strictly positive definite for all  $z \neq -1$  such that  $|z| = 1$ . Then the canonical spectral factor  $\mathbf{A}(z)$  of  $\mathbf{C}(z)$  defines a subdivision scheme  $S_{\mathcal{A}}$  which converges in  $L_2^{r \times r}(\mathbb{R})$  to an orthonormal matrix scaling function  $\mathbf{F}$ .*

As to the construction of such function, in [2] a simple procedure is given based on a modified *spectral factorization* of the *para-hermitian matrix*  $2\mathbf{C}(z)$ , which takes into account the presence of some of its zeros on the unitary circle.

We can now associate a matrix wavelet to any orthogonal MCMRA in the standard way. A matrix function  $\mathbf{G} \in V_1$  is called a *multichannel wavelet* for the orthogonal MCMRA if:

1.  $W_j := V_{j+1} \ominus V_j = \{\mathbf{G} * c(2^j \cdot) : c \in \ell_2^r(\mathbb{Z})\}$ ,  $j \in \mathbb{Z}$ ;
2.  $\mathbf{G}$  is orthonormal.

As to the existence of multichannel wavelets, we recall the following result:

**Theorem 3.** [1] Suppose that the matrix scaling function  $\mathbf{F} \in L_2^{r \times r}(\mathbb{R})$  has orthonormal integer translates. Then there exists an orthonormal wavelet  $\mathbf{G} \in L_2^{r \times r}(\mathbb{R})$  satisfying the two-scale relation

$$\mathbf{G} = \sum_{j \in \mathbb{Z}} \mathbf{F}(2 \cdot - j) \mathbf{B}_j \quad (7)$$

for a suitable mask  $\mathcal{B} = (\mathbf{B}_j : j \in \mathbb{Z})$ .

Observe that, as in the scalar case, due to the full rank properties of  $\mathbf{F}$ , the symbol of the multichannel wavelet  $\mathbf{G}$  must possess at least one factor  $(z - 1)$ . This is equivalent to say that the multichannel wavelet has at least one vanishing moment.

It is easy to show that the symbols of an orthonormal matrix function  $\mathbf{F}$  and of the corresponding multichannel wavelet  $\mathbf{G}$  satisfy the *orthogonality (quadrature mirror filter (QMF)) conditions*

$$\begin{aligned} \mathbf{A}^\sharp(z) \mathbf{A}(z) + \mathbf{A}^\sharp(-z) \mathbf{A}(-z) &= 4\mathbf{I}, \quad |z| = 1, \\ \mathbf{A}^\sharp(z) \mathbf{B}(z) + \mathbf{A}^\sharp(-z) \mathbf{B}(-z) &= \mathbf{0}, \quad |z| = 1, \\ \mathbf{B}^\sharp(z) \mathbf{B}(z) + \mathbf{B}^\sharp(-z) \mathbf{B}(-z) &= 4\mathbf{I}, \quad |z| = 1, \end{aligned}$$

where  $\mathbf{A}^\sharp(z) := \mathbf{A}^T(z^{-1})$  which means  $\mathbf{A}^\sharp(z) := \mathbf{A}^H(z)$  whenever  $|z| = 1$ .

The QMF can be written in a concise form by the condition  $\mathbf{U}^\sharp(z) \mathbf{U}(z) = \mathbf{I}$  on the block matrix

$$\mathbf{U}(z) := \frac{1}{2} \begin{pmatrix} \mathbf{A}(z) & \mathbf{B}(z) \\ \mathbf{A}(-z) & \mathbf{B}(-z) \end{pmatrix}, \quad \text{where } \mathbf{U}^\sharp(z) := \frac{1}{2} \begin{pmatrix} \mathbf{A}^\sharp(z) & \mathbf{A}^\sharp(-z) \\ \mathbf{B}^\sharp(z) & \mathbf{B}^\sharp(-z) \end{pmatrix},$$

(which is the condition of being an unitary matrix for  $|z| = 1$ ).

Note that, given an orthogonal symbol  $\mathbf{A}(z)$ , the alternating flip trick, used to construct the wavelet symbol in the scalar case, does not work in this context. Indeed, the symbol  $\mathbf{B}(z) = z\mathbf{A}^\sharp(-z)$  does not verify the QMF equations unless  $\mathbf{A}(z)$  and  $\mathbf{A}(-z)$  commute.

On the other hand, the orthogonality conditions can be written in terms of the subsymbols  $\mathbf{A}_0(z)$ ,  $\mathbf{A}_1(z)$  and  $\mathbf{B}_0(z)$ ,  $\mathbf{B}_1(z)$ . In fact, from (4) we see that the matrix  $\mathbf{V}(z)$

$$\mathbf{V}(z) := \begin{pmatrix} \mathbf{A}_0(z^2) & \mathbf{B}_0(z^2) \\ \mathbf{A}_1(z^2) & \mathbf{B}_1(z^2) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{z} & -\frac{1}{z} \end{pmatrix} \mathbf{U}(z)$$

is such that  $\mathbf{V}^\sharp(z) \mathbf{V}(z) = 2\mathbf{I}$ . Thus, the QMF equations take the equivalent form

$$\begin{aligned} \mathbf{A}_0^\sharp(z^2) \mathbf{A}_0(z^2) + \mathbf{A}_1^\sharp(z^2) \mathbf{A}_1(z^2) &= 2\mathbf{I}, \quad |z| = 1, \\ \mathbf{A}_0^\sharp(z^2) \mathbf{B}_0(z^2) + \mathbf{A}_1^\sharp(z^2) \mathbf{B}_1(z^2) &= \mathbf{0}, \quad |z| = 1, \\ \mathbf{B}_0^\sharp(z^2) \mathbf{B}_0(z^2) + \mathbf{B}_1^\sharp(z^2) \mathbf{B}_1(z^2) &= 2\mathbf{I}, \quad |z| = 1. \end{aligned} \quad (8)$$



Suppose now that we are given an orthogonal MCMRA generated by some orthonormal matrix scaling function  $\mathbf{F}$ , with  $\mathbf{G}$  as the corresponding multichannel wavelet. Let us consider any vector-valued function  $\mathbf{h} \in L_2^r(\mathbb{R})$ . From the nesting properties of the spaces  $\{V_j\}$  and  $\{W_j\}$ , it follows that the approximation  $P_\ell \mathbf{h}$  of  $\mathbf{h}$  in the space  $V_\ell$ ,  $\ell \in \mathbb{Z}$ , can be found in terms of the following *multichannel wavelet decomposition*:

$$P_\ell \mathbf{h} = P_{\ell-L} \mathbf{h} + Q_{\ell-1} \mathbf{h} + Q_{\ell-2} \mathbf{h} + \dots + Q_{\ell-L} \mathbf{h},$$

where  $L > 0$  and  $P_{\ell-L} \mathbf{h}$ ,  $Q_{\ell-j} \mathbf{h}$ ,  $j = 1, \dots, L$ , represent the orthogonal projections of  $\mathbf{h}$  to the spaces  $V_{\ell-L}$ ,  $W_{\ell-j}$ ,  $j = 1, \dots, L$ , respectively. Analogously to the scalar case, we can derive a fast algorithm which allows to compute all the projections by means of a recursive scheme. In fact we have that, if  $P_j \mathbf{h} \in V_j$  then

$$P_j \mathbf{h} = \sum_{k \in \mathbb{Z}} \mathbf{F}(2^j \cdot -k) \mathbf{c}_k^{(j)}$$

where

$$\mathbf{c}_k^{(j)} = \langle \mathbf{F}(2^j \cdot -k), \mathbf{h} \rangle = \int \mathbf{F}^H(2^j x - k) \mathbf{h}(x) dx.$$

To compute the vector coefficient sequence  $\mathbf{c}^{(j-1)} \in \ell_2^r(\mathbb{Z})$  connected to the representation of  $\mathbf{h}$  in the space  $V_{j-1}$ , we make use of the refinement equation and get

$$\begin{aligned} \mathbf{c}_k^{(j-1)} &= \langle \mathbf{F}(2^{j-1} \cdot -k), \mathbf{h} \rangle = \langle \sum_{n \in \mathbb{Z}} \mathbf{F}(2(2^{j-1} \cdot -k) - n) \mathbf{A}_n, \mathbf{h} \rangle \\ &= \sum_{n \in \mathbb{Z}} \mathbf{A}_{n-2k}^T \langle \mathbf{F}(2^j \cdot -n), \mathbf{h} \rangle = \sum_{n \in \mathbb{Z}} \mathbf{A}_{n-2k}^T \mathbf{c}_n^j, \quad k \in \mathbb{Z}. \end{aligned}$$

In the same way, using (7), the wavelet vector coefficients sequence  $\mathbf{d}^{(j-1)} \in \ell_2^r(\mathbb{Z})$  connected to the representation of  $\mathbf{h}$  in the space  $W_{j-1}$  is obtained as

$$\mathbf{d}_k^{(j-1)} = \langle \mathbf{G}(2^{j-1} \cdot -k), \mathbf{h} \rangle = \sum_{n \in \mathbb{Z}} \mathbf{B}_{n-2k}^T \mathbf{c}_n^j, \quad k \in \mathbb{Z}.$$

In summary, assuming  $\ell = 0$ , that is  $V_0$  as the initial space of our representation, the *vector decomposition formula* up to the level  $L > 0$  reads as

$$\mathbf{c}_k^{(j-1)} = \sum_{n \in \mathbb{Z}} \mathbf{A}_{n-2k}^T \mathbf{c}_n^j, \quad \mathbf{d}_k^{(j-1)} = \sum_{n \in \mathbb{Z}} \mathbf{B}_{n-2k}^T \mathbf{c}_n^j, \quad k \in \mathbb{Z}, j = 0, \dots, L.$$

Conversely, given the projections  $P_j \mathbf{h}$  and  $Q_j \mathbf{h}$ , the vector coefficient sequence  $\mathbf{c}^{(j+1)}$  connected to the representation of  $\mathbf{h}$  in the space  $V_{j+1} = V_j \oplus W_j$  is obtained by considering that

$$P_{j+1} \mathbf{h} = \sum_{k \in \mathbb{Z}} \mathbf{F}(2^{j+1} \cdot -k) \mathbf{c}_k^{(j+1)} \quad (9)$$

and, on the other hand,

$$P_{j+1}\mathbf{h} = P_j\mathbf{h} + Q_j\mathbf{h} = \sum_{n \in \mathbb{Z}} \mathbf{F}(2^j \cdot -n) \mathbf{c}_n^{(j)} + \sum_{n \in \mathbb{Z}} \mathbf{G}(2^j \cdot -n) \mathbf{d}_n^{(j)}.$$

By invoking again the refinement equation on the right-hand side of the previous expression, we have that

$$P_{j+1}\mathbf{h} = \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \mathbf{F}(2^{j+1} - k) \mathbf{A}_{k-2n} \mathbf{c}_n^{(j)} + \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \mathbf{G}(2^{j+1} - n) \mathbf{B}_{k-2n} \mathbf{d}_n^{(j)}. \quad (10)$$

A comparison between the two expressions (9) and (10) gives the *vector reconstruction formula*:

$$\mathbf{c}_k^{(j+1)} = \sum_{n \in \mathbb{Z}} \left( \mathbf{A}_{k-2n} \mathbf{c}_n^{(j)} + \mathbf{B}_{k-2n} \mathbf{d}_n^{(j)} \right), \quad k \in \mathbb{Z}, j = -L, \dots, 0.$$

For similar matrix wavelet decomposition and reconstruction schemes see [14].

## 5. Multichannel wavelet construction

The equation  $\mathbf{A}_0^\sharp(z) \mathbf{A}_0(z) + \mathbf{A}_1^\sharp(z) \mathbf{A}_1(z) = 2\mathbf{I}$  is the starting point of the procedure that we propose to construct the symbol  $\mathbf{B}(z)$  of the multichannel wavelet. More in detail, the procedure derives the matrix Laurent polynomials  $\mathbf{B}_0(z)$  and  $\mathbf{B}_1(z)$ , and thus  $\mathbf{B}(z) = \mathbf{B}_0(z^2) + z\mathbf{B}_1(z^2)$ .

In the simple situation where  $\mathbf{A}_0(z)$  and  $\mathbf{A}_1(z)$  are diagonal symbols, the multichannel wavelet subsymbols  $\mathbf{B}_0(z)$  and  $\mathbf{B}_1(z)$  are constructed by the repeated application of a scalar procedure: for each couple  $(a_0(z))_{ii}, (a_1(z))_{ii}$ ,  $i = 1, \dots, r$ , the two Laurent polynomials  $(b_0(z))_{ii}, (b_1(z))_{ii}$  solution of the Bezout identity are derived (see Lemma 1).

In the general situation, the strategy is based on the existence of two matrix symbols  $\mathbf{D}_0(z)$ ,  $\mathbf{D}_1(z)$  such that the positive definite Hermitian matrix

$$\mathbf{G}(z) =: \begin{pmatrix} \mathbf{A}_0(z) & \mathbf{D}_0^\sharp(z) \\ \mathbf{A}_1(z) & \mathbf{D}_1^\sharp(z) \end{pmatrix} \quad (11)$$

satisfies

$$\det \mathbf{G}(z) = 1, \quad \mathbf{G}^\sharp(z) \mathbf{G}(z) = \begin{pmatrix} 2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_0(z) \mathbf{D}_0^\sharp(z) + \mathbf{D}_1(z) \mathbf{D}_1^\sharp(z) \end{pmatrix}, \quad (12)$$

as proved in the following proposition.

**Proposition 3.** *Let  $\mathbf{A}_0(z)$  and  $\mathbf{A}_1(z)$  be such that  $\mathbf{A}_0^\sharp(z) \mathbf{A}_0(z) + \mathbf{A}_1^\sharp(z) \mathbf{A}_1(z) = 2\mathbf{I}$ . Suppose that there exist two matrix symbols  $\mathbf{D}_0(z)$ ,  $\mathbf{D}_1(z)$  such that the positive definite Hermitian matrix  $\mathbf{G}(z)$  in (11) has the properties (12). Let  $\mathbf{K}(z)$  be the spectral factor of  $\mathbf{D}(z) := 2 \left( \mathbf{D}_0(z) \mathbf{D}_0^\sharp(z) + \mathbf{D}_1(z) \mathbf{D}_1^\sharp(z) \right)$ . Then the matrix symbols*

$$\mathbf{B}_0(z) = 2\mathbf{D}_0^\sharp(z) \left( \mathbf{K}^\sharp(z) \right)^{-1}, \quad \mathbf{B}_1(z) = 2\mathbf{D}_1^\sharp(z) \left( \mathbf{K}^\sharp(z) \right)^{-1}, \quad (13)$$

*satisfy the two last equations in (8).*

*Proof.* Since  $\mathbf{D}(z)$  is positive definite with determinant equal to 1, its spectral factor  $\mathbf{K}(z)$  exists and satisfies  $\mathbf{D}(z) = \mathbf{K}(z)\mathbf{K}^\sharp(z)$ . Therefore, if we substitute (13) into

$$\mathbf{A}_0^\sharp(z)\mathbf{D}_0^\sharp(z) + \mathbf{A}_1^\sharp(z)\mathbf{D}_1^\sharp(z) = \mathbf{0}$$

we get

$$\mathbf{A}_0^\sharp(z^2)\mathbf{B}_0(z^2) + \mathbf{A}_1^\sharp(z^2)\mathbf{B}_1(z^2) = \mathbf{0}, \quad |z| = 1,$$

which is the second equation in (8). Finally, since

$$(\mathbf{K}(z))^{-1} \mathbf{D}(z) (\mathbf{K}^\sharp(z))^{-1} = \mathbf{I},$$

by substituting (13) into it we end up with

$$\mathbf{B}_0^\sharp(z^2)\mathbf{B}_0(z^2) + \mathbf{B}_1^\sharp(z^2)\mathbf{B}_1(z^2) = 2\mathbf{I}, \quad |z| = 1$$

which is the last equation in (8).  $\square$

For the proof of the existence of the Hermitian matrix  $\mathbf{G}(z)$  we refer the reader to [11]. Its actual construction can be carried out as explained in the following subsections.

#### 5.1. Some matrix completion results

We start by recalling a result about Bezout identities

**Theorem 4.** [10] *For any pair of Laurent polynomials  $a_1(z)$   $a_2(z)$  the Bezout identity*

$$a_1(z)b_1(z) + a_2(z)b_2(z) = 1 \tag{14}$$

*is satisfied by a pair  $b_1(z)$   $b_2(z)$  of Laurent polynomials if and only if  $a_1(z)$  and  $a_2(z)$  have no common zeros. Moreover, given one particular pair of solutions  $b_1^*(z)$ ,  $b_2^*(z)$  the set of all solutions of (14) is of the form*

$$b_1(z) = b_1^*(z) + p(z)a_2(z), \quad b_2(z) = b_2^*(z) - p(z)a_1(z)$$

*where  $p(z)$  is any Laurent polynomial.*

As to the common zeros of the subsymbols, the following result holds true.

**Lemma 1.** *The subsymbols associated with the diagonal entries of an orthogonal symbol  $\mathbf{A}(z)$ , that is the subsymbols  $(a_0(z))_{ii}$ ,  $(a_1(z))_{ii}$ ,  $i = 1, \dots, r$ , have no common zeros.*

*Proof.* The result follows from the first equation in (8). In fact the existence of a common zero for  $(a_0(z))_{ii}$ ,  $(a_1(z))_{ii}$ ,  $i = 1, \dots, r$  contradicts

$$(a_0(z^{-1}))_{ii}(a_0(z))_{ii} + (a_1(z^{-1}))_{ii}(a_1(z))_{ii} = 2, \quad i = 1, \dots, r,$$

a relation satisfied by the diagonal symbols.  $\square$   $\square$

We continue with a "completion" result.

**Theorem 5.** *Let  $a_i(z)$ ,  $i = 1, \dots, n$ , Laurent polynomials with  $n \geq 2$  with no common zeros. Then there exists a  $n \times n$  matrix Laurent polynomial  $\mathbf{P}(z)$  whose first row is  $(a_1(z), \dots, a_n(z))$  such that*

$$\det \mathbf{P}(z) = 1, \quad z \in \mathbb{C} \setminus \{0\}.$$

*Proof.* The proof is by induction. Let us start with  $n = 2$ . For the row vector  $(a_1(z), a_2(z))$  we construct the matrix  $\mathbf{P}(z) = \begin{pmatrix} a_1(z) & a_2(z) \\ -b_1(z) & b_2(z) \end{pmatrix}$  with  $b_i(z)$ ,  $i = 1, 2$  being solutions of the Bezout identity (14). Obviously, the matrix  $\mathbf{P}(z)$  is such that  $\det \mathbf{P}(z) = 1$ ,  $z \in \mathbb{C} \setminus \{0\}$ . Next, assuming that the theorem is true for  $n - 1$ , we prove it for  $n$ . Thus, given  $(a_1(z), \dots, a_n(z))$  we first construct  $\bar{\mathbf{P}}(z)$ ,  $\det \bar{\mathbf{P}}(z) = 1$ , with the first row given by  $(a_1(z), \dots, a_{n-1}(z))$  then we construct the block matrix

$$\mathbf{P}(z) := \begin{pmatrix} & & & & a_n(z) \\ & & & & 0 \\ & & & & \vdots \\ & & \bar{\mathbf{P}}(z) & & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

This completes the proof.  $\square$

With this result we are now able to describe a procedure for "completing" an  $m \times n$  matrix Laurent polynomial with  $m \geq n$  and  $\text{rank } \mathbf{A}(z) = m$ , by constructing an  $n \times n$  matrix Laurent polynomial  $\mathbf{P}(z)$ , whose first  $m$  rows agree with those of  $\mathbf{A}(z)$ , such that

$$\det \mathbf{P}(z) = 1, \quad z \in \mathbb{C} \setminus \{0\}.$$

The existence of such polynomial has been formerly proved in [11].

We start by discussing how to write a procedure providing, for a given Laurent polynomial vector  $\mathbf{a}_n(z)$  of length  $n \geq 2$ , a Laurent polynomial matrix  $\mathbf{P}(z)$  having the first row given by  $\mathbf{a}_n(z) = (a_1(z), \dots, a_n(z))$  and satisfying  $\det \mathbf{P}(z) = 1$ ,  $z \in \mathbb{C} \setminus \{0\}$ .

For  $a_1(z), a_2(z)$ , let  $[b_1(z), b_2(z)] = \text{Bezout}(a_1(z), a_2(z))$  be the procedure in Matlab-like notation providing the two Laurent polynomials  $b_1(z), b_2(z)$  solution of the Bezout identity.

Next, let  $[\mathbf{P}_{n \times n}(z)] = \text{Basic\_completion}(\mathbf{a}_n(z))$  be the following (recursive) procedure that, taking the vector  $\mathbf{a}(z)$  as input, produces the matrix  $\mathbf{P}_{n \times n}(z)$  as output.

- $[\mathbf{P}_{n \times n}(z)] = \text{Basic\_completion}(\mathbf{a}_n(z))$

If  $n = 2$  let  $[b_1(z), b_2(z)] = \text{Bezout}(a_1(z), a_2(z))$  two Laurent polynomials solution of the Bezout identity. Set  $\mathbf{P}_{2 \times 2}(z) := \begin{pmatrix} a_1(z) & a_2(z) \\ -b_1(z) & b_2(z) \end{pmatrix}$ ;  
else ( $n > 2$ ) use  $[\mathbf{P}_{(n-1) \times (n-1)}(z)] = \text{Basic\_completion}(\mathbf{a}_{n-1}(z))$ .

Then with  $\mathbf{e}_{n-1} := (0, \dots, 1)^T \in \mathbb{R}^{n-1}$ , define

$$\mathbf{P}_{n \times n}(z) := \begin{pmatrix} \mathbf{P}_{(n-1) \times (n-1)}(z) & a_n(z) \\ \mathbf{0}_{1 \times (n-1)} & \mathbf{e}_{n-1} \end{pmatrix}.$$

We continue by describing the (recursive) procedure  $[\mathbf{P}_{m \times m}] = \text{Completion}(\mathbf{A}_{n \times m})$ . This procedure, for a given matrix Laurent polynomial  $\mathbf{A}_{n \times m}(z)$  with  $m > n$ , constructs the Laurent polynomial matrix  $\mathbf{P}(z)$  whose  $n$  first rows agree with those of  $\mathbf{A}_{n \times m}$  and such that  $\det \mathbf{P}(z) = 1$ ,  $z \in \mathbb{C} \setminus \{0\}$ .

•  $[\mathbf{P}_{m \times m}(z)] = \text{Completion}(\mathbf{A}_{n \times m}(z))$

1. Take  $\bar{\mathbf{A}}_{(n-1) \times m}(z)$  the sub-matrix of  $\mathbf{A}_{n \times m}(z)$  made of its first  $(n-1)$  rows;
2. If  $n = 2$  construct  $[\bar{\mathbf{P}}_{m \times m}(z)] = \text{Basic\_completion}(\bar{\mathbf{A}}_{1 \times m}(z))$   
else ( $n > 2$ ) construct  $[\bar{\mathbf{P}}_{m \times m}(z)] = \text{Completion}(\bar{\mathbf{A}}_{(n-1) \times m}(z))$ ;
3. Compute the matrix  $\mathbf{C}_{n \times m}(z) := \mathbf{A}_{n \times m}(z) \bar{\mathbf{P}}_{m \times m}^{-1}(z)$  having block structure

$$\mathbf{C}_{n \times m}(z) = \begin{pmatrix} \mathbf{I}_{(n-1) \times (n-1)} & \mathbf{0}_{(n-1) \times (r+1)} \\ \mathbf{c}_{1 \times (n-1)}(z) & \mathbf{d}_{1 \times (r+1)}(z) \end{pmatrix}$$

where  $r := m - n$ ;

4. Construct the  $(r+1) \times (r+1)$  matrix  
 $[\mathbf{D}_{(r+1) \times (r+1)}(z)] = \text{Basic\_completion}(\mathbf{d}_{1 \times (r+1)}(z))$ ;
5. Use  $\mathbf{D}_{(r+1) \times (r+1)}(z)$  to construct the  $m \times m$  block matrix

$$\mathbf{V}_{m \times m}(z) := \left( \begin{array}{c|c} \mathbf{I}_{(n-1) \times (n-1)}(z) & \mathbf{0}_{(n-1) \times (r+1)} \\ \hline \mathbf{c}_{1 \times (n-1)}(z) & \mathbf{D}_{(r+1) \times (r+1)}(z) \\ \hline \mathbf{0}_{r \times (n-1)} & \end{array} \right)$$

6. Set  $\mathbf{P}_{m \times m}(z) := \mathbf{V}_{m \times m}(z) \bar{\mathbf{P}}_{m \times m}(z)$ .

With the help of the two previous recursive procedures, we are able to sketch the algorithm for constructing multichannel wavelets.

### 5.2. The MCW construction algorithm

1. From  $\mathbf{A}(z) \in \ell_0^{r \times r}(\mathbb{Z})$  extract the sub-symbols  $\mathbf{A}_0(z)$  and  $\mathbf{A}_1(z)$ ;
2. If  $\mathbf{A}_0(z)$  and  $\mathbf{A}_1(z)$  are diagonal symbols construct  $\mathbf{B}_0(z)$  and  $\mathbf{B}_1(z)$  by the repeated application of the procedure  
 $[(b_0(z))_{ii}, (b_1(z))_{ii}] = \text{Bezout}((a_0(z))_{ii}, (a_1(z))_{ii})$ ,  $i = 1, \dots, r$   
Then go to step 10;
3. Otherwise construct the  $r \times 2r$  block matrix  $\tilde{\mathbf{A}}^\sharp(z) = \begin{pmatrix} \mathbf{A}_0^\sharp(z) & \mathbf{A}_1^\sharp(z) \end{pmatrix}$ ;
4. Use  $[\mathbf{L}(z)] = \text{Completion}(\tilde{\mathbf{A}}^\sharp(z))$  to construct the matrix

$$\mathbf{L}(z) =: \begin{pmatrix} \mathbf{A}_0^\sharp(z) & \mathbf{A}_1^\sharp(z) \\ \mathbf{C}_0(z) & \mathbf{C}_1(z) \end{pmatrix}$$

so that

$$\det \begin{pmatrix} \mathbf{A}_0(z) & \mathbf{C}_0^\sharp(z) \\ \mathbf{A}_1(z) & \mathbf{C}_1^\sharp(z) \end{pmatrix} = 1;$$

5. Compute the  $r \times r$  matrix  $\mathbf{R}(z) = \frac{1}{2} \left( \mathbf{A}_0^\sharp(z) \mathbf{C}_0^\sharp(z) + \mathbf{A}_1^\sharp(z) \mathbf{C}_1^\sharp(z) \right)$ ;
6. Compute the  $r \times r$  matrices  $\mathbf{D}_i(z) = -\mathbf{R}^\sharp(z) \mathbf{A}_i^\sharp(z) + \mathbf{C}_i(z)$ ,  $i = 0, 1$ ;
7. Compute the  $r \times r$  positive definite matrix

$$\mathbf{D}(z) = 2 \left( \mathbf{D}_0(z) \mathbf{D}_0^\sharp(z) + \mathbf{D}_1(z) \mathbf{D}_1^\sharp(z) \right);$$

8. Compute the spectral factorization of  $\mathbf{D}(z)$  such that

$$\mathbf{D}(z) = \mathbf{K}(z) \mathbf{K}^\sharp(z),$$

for example using the algorithm described in [9];

9. Set  $\mathbf{E}(z) = \mathbf{K}^{-1}(z)$  and construct the  $r \times r$  matrices

$$\mathbf{B}_i(z) = 2 \mathbf{D}_i^\sharp(z) \mathbf{E}^\sharp(z), \quad i = 0, 1;$$

10. Construct the wavelet symbol

$$\mathbf{B}(z) = \mathbf{B}_0(z^2) + z \mathbf{B}_1(z^2).$$

Two remarks are worth to be made:

- For a proof of the correctness of all steps we refer to [11, Section 6]. Nevertheless, since [11] deals with rank-1 matrix functions, the above construction has to be considered as an extension to the full rank case;
- The algorithm produces correct results even in case of diagonal subsymbols  $\mathbf{A}_0(z)$  and  $\mathbf{A}_1(z)$ . Though, to end up with exactly diagonal subsymbols  $\mathbf{B}_0(z)$  and  $\mathbf{B}_1(z)$  we may need to multiply the result of the MCW algorithm by a suitable permutation matrix  $\mathbf{J}$ .

## 6. Numerical Examples

The aim of this section is to consider two orthonormal full rank refinable matrix functions with  $r = 2$  and  $r = 3$ , respectively and to use the MCW algorithm described in the subsection 5.2 to construct the corresponding multichannel wavelets. In particular, while for the case  $r = 2$  the positive definite full rank symbol that we start with is the one considered in [2], the case  $r = 3$  is here derived from scratch.

### 6.1. A two-channel example

Let  $\mathbf{C}(z)$  be the symbol of the interpolatory full rank vector subdivision scheme first given in [2]

$$\mathbf{C}(z) = \begin{pmatrix} c_{11}(z) & c_{12}(z) \\ c_{12}(z^{-1}) & c_{22}(z) \end{pmatrix}$$

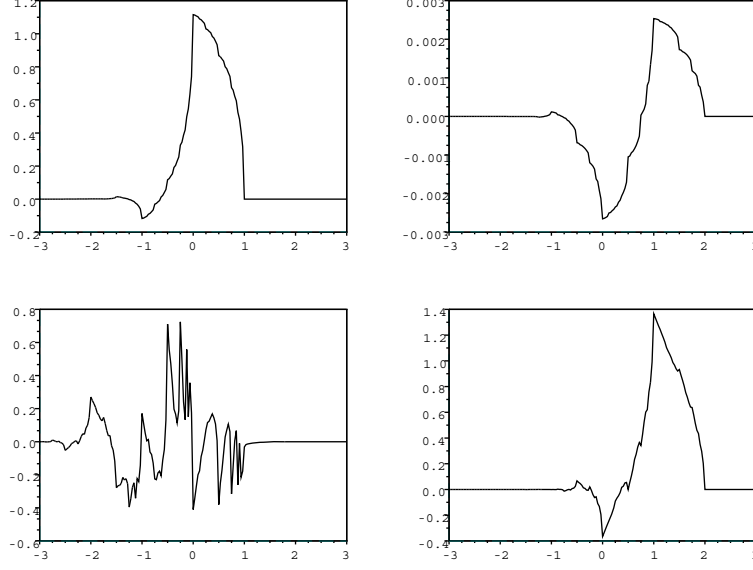


Figure 1: Orthonormal matrix scaling function with  $r = 2$

with

$$\begin{aligned}
c_{11}(z) &= \frac{1}{2} \frac{(z+1)^2}{z} \quad (\text{linear B-spline symbol}), \\
c_{22}(z) &= -\frac{1}{16} \frac{(z^2 - 4z + 1)(z+1)^4}{z^3} \quad (\text{4-point scheme symbol}), \\
c_{12}(z) &= \lambda z (z^2 - 1)^3,
\end{aligned}$$

satisfying  $\mathbf{C}(1) = 2\mathbf{I}$ ,  $\mathbf{C}(z) + \mathbf{C}(-z) = 2\mathbf{I}$ . In order to assure positive definiteness, the parameter  $\lambda$  is taken in the interval  $(-\frac{1}{32}\sqrt{3}, \frac{1}{32}\sqrt{3})$ . Fixing, for example,  $\lambda = \frac{1}{20}$ , the symbol  $\mathbf{A}(z)$  of the associated full rank orthonormal refinable function constructed with the algorithm given in [2] has elements

$$\begin{aligned}
a_{11}(z) &= 1.081604742 + 0.7776021479z + 0.2165957837z^{-1} - 0.08257171989z^{-2} \\
&\quad + 0.004835070286z^{-3} + 0.0009670018466z^{-4} + 0.0009670219776z^{-5}, \\
a_{12}(z) &= -0.001208777472 + 0.001208777472z^2 + 0.001208777472z - 0.001208777472z^{-1}, \\
a_{21}(z) &= -0.02117594497 - 0.005136018632z - 0.07331687185z^{-1} + 0.2747671662z^{-2} \\
&\quad - 0.0679559177z^{-3} - 0.2535912212z^{-4} + 0.1464088082z^{-5}, \\
a_{22}(z) &= 0.3169890016 + 0.6830110222z^2 + 1.183011034z - 0.1830110103z^{-1}.
\end{aligned}$$

The plot of the corresponding matrix scaling function is given in Fig. 1

Using the MCW Algorithm we find that the symbol  $\mathbf{B}(z)$  of the corresponding orthonormal two-channel wavelet has the coefficients  $\mathbf{B}_k$ ,  $k = -2, \dots, 7$ , given in Table 1. The plot of the wavelet is represented in Fig. 2.

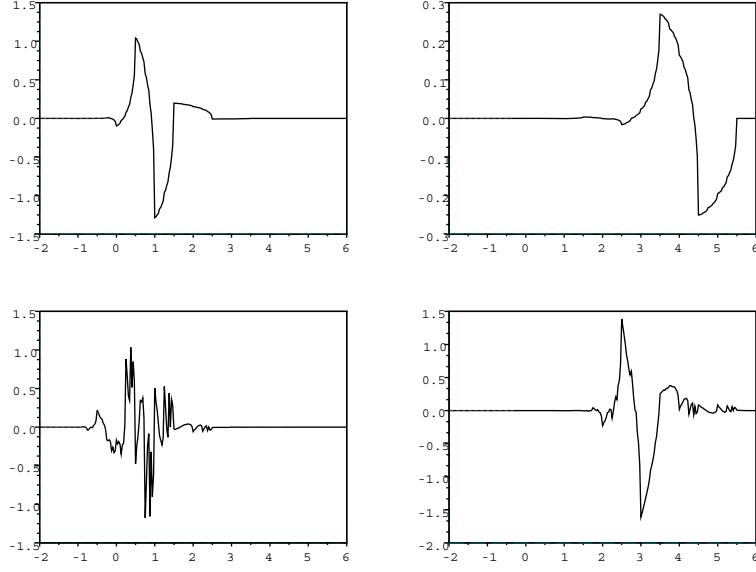


Figure 2: Orthonormal 2-channel wavelet

Table 1: Coefficients of the 2-channel orthonormal wavelet

$k$	$B_k$	$k$	$B_k$
-2	$\begin{pmatrix} 0.8140199 & 0 \\ -0.0014406 & 0 \end{pmatrix}$	3	$\begin{pmatrix} -0.0044401 & 0.0519219 \\ -0.0000359 & 0.331059 \end{pmatrix}$
-1	$\begin{pmatrix} -1.1322992 & 0 \\ 0 & 0 \end{pmatrix}$	4	$\begin{pmatrix} 0.0009821 & 0.2546764 \\ -0.0000192 & 0.1253084 \end{pmatrix}$
0	$\begin{pmatrix} 0.1915220 & 0.0032939 \\ 0.0037742 & -0.0000058 \end{pmatrix}$	5	$\begin{pmatrix} 0.0007061 & 0.1209272 \\ -0.0000047 & 0.0153323 \end{pmatrix}$
1	$\begin{pmatrix} 0.1360351 & -0.0002724 \\ -0.0018743 & 0.6524543 \end{pmatrix}$	6	$\begin{pmatrix} 0 & -0.2427089 \\ 0 & 0.0047518 \end{pmatrix}$
2	$\begin{pmatrix} -0.0065259 & -0.0133465 \\ -0.0003995 & -1.1300526 \end{pmatrix}$	7	$\begin{pmatrix} 0 & -0.1744916 \\ 0 & 0.0011525 \end{pmatrix}$

### 6.2. A three-channel example

We now construct a  $3 \times 3$  positive definite parahermitian interpolatory symbol

$$C(z) = \begin{pmatrix} c_{11}(z) & c_{12}(z) & c_{13}(z) \\ c_{12}(z^{-1}) & c_{22}(z) & c_{23}(z) \\ c_{13}(z^{-1}) & c_{23}(z^{-1}) & c_{33}(z) \end{pmatrix}$$



Table 2: Coefficients of the 3-channel orthonormal matrix scaling function

$k$	$\mathbf{A}_k$	$k$	$\mathbf{A}_k$
-8	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.04 & 0.03125 & 0 \end{pmatrix}$	-2	$\begin{pmatrix} -0.0715410 & 0.0216501 & 0 \\ 0.2697914 & -0.0206268 & 0 \\ -0.04 & -0.03125 & 0 \end{pmatrix}$
-7	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.04 & -0.03125 & 0 \end{pmatrix}$	-1	$\begin{pmatrix} 0.2874359 & 0.0253824 & 0 \\ -0.0756559 & -0.1655344 & 0 \\ 0.04 & 0.03125 & 0 \end{pmatrix}$
-6	$\begin{pmatrix} -0.0022120 & -0.0017281 & 0 \\ 0.0000731 & 0.0000571 & 0 \\ -0.12 & -0.09375 & 0 \end{pmatrix}$	0	$\begin{pmatrix} 1.0740925 & -0.0422890 & 0 \\ -0.0129914 & 0.3501563 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
-5	$\begin{pmatrix} 0.0087982 & 0.0060117 & 0 \\ 0.1481828 & -0.0017879 & 0 \\ 0.12 & 0.09375 & 0 \end{pmatrix}$	1	$\begin{pmatrix} 0.7231598 & -0.0073621 & 0 \\ -0.0053020 & 1.1683135 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
-4	$\begin{pmatrix} -0.0003396 & 0.0004183 & 0 \\ -0.2568731 & 0.0059962 & 0 \\ 0.12 & 0.09375 & 0 \end{pmatrix}$	2	$\begin{pmatrix} 0 & 0.0219488 & 0 \\ 0 & 0.6644173 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
-3	$\begin{pmatrix} -0.0193938 & -0.0240319 & 0 \\ -0.0672249 & -0.0009911 & 0 \\ -0.12 & -0.09375 & 0 \end{pmatrix}$		

satisfying  $\mathbf{C}(1) = 2\mathbf{I}$ ,  $\mathbf{C}(z) + \mathbf{C}(-z) = 2\mathbf{I}$ . It means that, as in the previous example, the diagonal elements should be the symbols of interpolatory scalar schemes, with factor  $(z+1)$  of order  $m_1, m_2, m_3$ , respectively. The off-diagonals should certainly contain a  $(z^2-1)$  factor and must satisfy  $c_{ij}(z) = -c_{ij}(-z)$  (see [4]). We take on the diagonal the Deslaurier-Dubuc filters with  $m_1 = 2, m_2 = 4, m_3 = 2$ , that is:

$$c_{11}(z) = \frac{1}{2} \frac{(z+1)^2}{z}, \quad c_{22}(z) = -\frac{1}{16} \frac{(z^2-4z+1)(z+1)^4}{z^3}, \quad c_{33}(z) = c_{11}(z).$$

Since  $(z+1)^2$  is a common factor, repeating the consideration done in [2], we require the following symbols on the off-diagonal

$$c_{12}(z) = \lambda_1 z (z^2-1)^3, \quad c_{13}(z) = \lambda_2 z (z^2-1)^4, \quad c_{23}(z) = \lambda_3 z (z^2-1)^4.$$

Table 3: Coefficients of the 3-channel orthonormal matrix wavelet

$k$	$\mathbf{B}_k$	$k$	$\mathbf{B}_k$
-2	$\begin{pmatrix} -0.0031659 & 0 & 0 \\ 0.0001046 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	5	$\begin{pmatrix} 0.1684248 & 0.0212803 & 0.0595696 \\ -0.0017438 & 0.0229928 & 0.1406681 \\ -0.0001092 & -0.0004847 & -0.0160288 \end{pmatrix}$
-1	$\begin{pmatrix} 0 & 0 & 0 \\ -0.6416072 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	6	$\begin{pmatrix} -0.0017816 & -0.0078392 & -0.1305600 \\ -0.0009269 & 0.0105339 & 0.0451634 \\ 0 & 0 & -0.0026508 \end{pmatrix}$
0	$\begin{pmatrix} 0.0297628 & -0.7705220 & -0.0001305 \\ 1.1269644 & 0.0254539 & 0 \\ 0.0367180 & 0.0169580 & -0.9720037 \end{pmatrix}$	7	$\begin{pmatrix} -0.0012167 & -0.0056664 & -0.0773112 \\ -0.0015390 & -0.0062807 & -0.1222042 \\ 0 & 0 & 0.0026508 \end{pmatrix}$
1	$\begin{pmatrix} -0.0940031 & 1.1447719 & 0 \\ -0.3500526 & -0.0169113 & -0.0264419 \\ -0.0367180 & -0.0169580 & 0.9720037 \end{pmatrix}$	8	$\begin{pmatrix} -0.0000291 & -0.0001188 & 0.0406917 \\ -0.0008803 & -0.0035956 & -0.0543552 \\ 0 & 0 & 0 \end{pmatrix}$
2	$\begin{pmatrix} -0.2583832 & -0.2560664 & -0.0751903 \\ -0.1287475 & -0.0115762 & 0.0489688 \\ -0.0096495 & -0.0033517 & -0.0409056 \end{pmatrix}$	9	$\begin{pmatrix} 0 & 0 & 0.0295229 \\ 0 & 0 & 0.0373432 \\ 0 & 0 & 0 \end{pmatrix}$
3	$\begin{pmatrix} -0.0908757 & -0.1606346 & 0.0018325 \\ -0.0045318 & -0.0170932 & -0.0567853 \\ 0.0096495 & 0.0033517 & 0.0409056 \end{pmatrix}$	10	$\begin{pmatrix} 0 & 0 & 0.0007057 \\ 0 & 0 & 0.0213611 \\ 0 & 0 & 0 \end{pmatrix}$
4	$\begin{pmatrix} 0.2512676 & 0.0347952 & 0.1508696 \\ 0.0029602 & -0.0035235 & -0.0337223 \\ 0.0001092 & 0.0004847 & 0.0160288 \end{pmatrix}$		

The following parameter choice

$$\lambda_1 = 1/20, \lambda_2 = 1/50, \lambda_3 = 1/64$$

gives positive definiteness. In this case, the canonical spectral factor  $\mathbf{A}(z)$ , symbol of the orthonormal refinable function  $\mathbf{F}$ , has the coefficients given in Table 2. The plot of the corresponding scaling function is represented in Fig. 3.

The MCW algorithm applied to this symbol, gives the orthonormal wavelet whose symbol coefficients are given in Table 3 and whose plot is represented in Fig. 4.

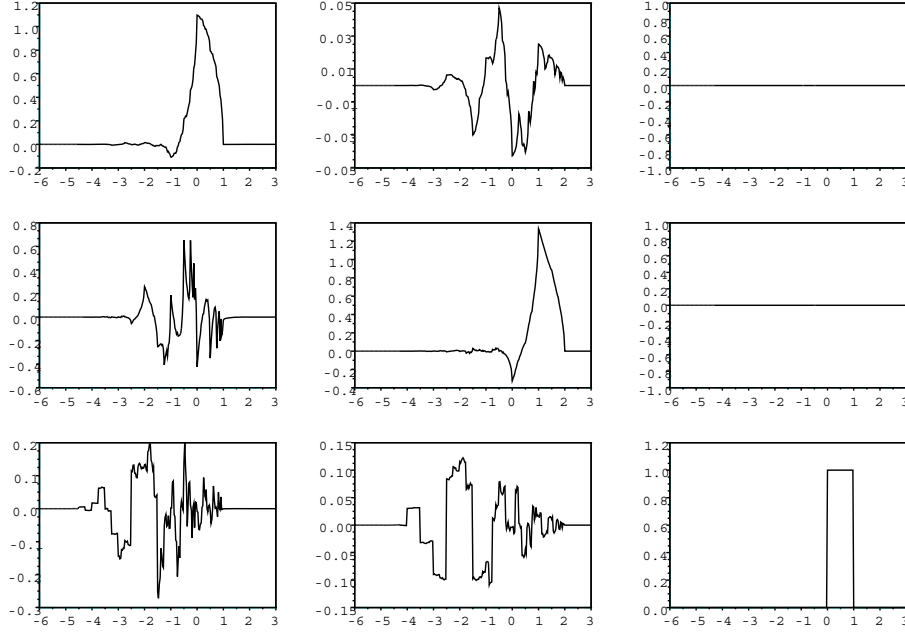


Figure 3: Orthonormal matrix scaling function with  $r = 3$

## 7. Conclusions

This paper discusses a way to construct orthonormal multichannel wavelets from convergent full rank orthogonal symbols. As to our knowledge, the explicit algorithm for multichannel wavelet construction (the MCW construction algorithm) here derived, based on the strategy for rank-1 filters given in [11], is the first algorithm proposed so far in the literature.

As already pointed out, multichannel wavelets provide an effective tool for the analysis of multichannel signals, that is vector-valued signals whose components come from different sources with possible intrinsic correlations, for example seismic waves, brain activity (EEG/MEG) data, financial time series, color images. Many multichannel signals exhibit a high correlation which can be revealed and exploited by a MCW analysis, with filters suitably tailored to the specific data and application. Future investigations include the construction of "data-adapted" MCW bases and their application in many problems where vector-valued signals have to be processed for compression, denoising, etc.

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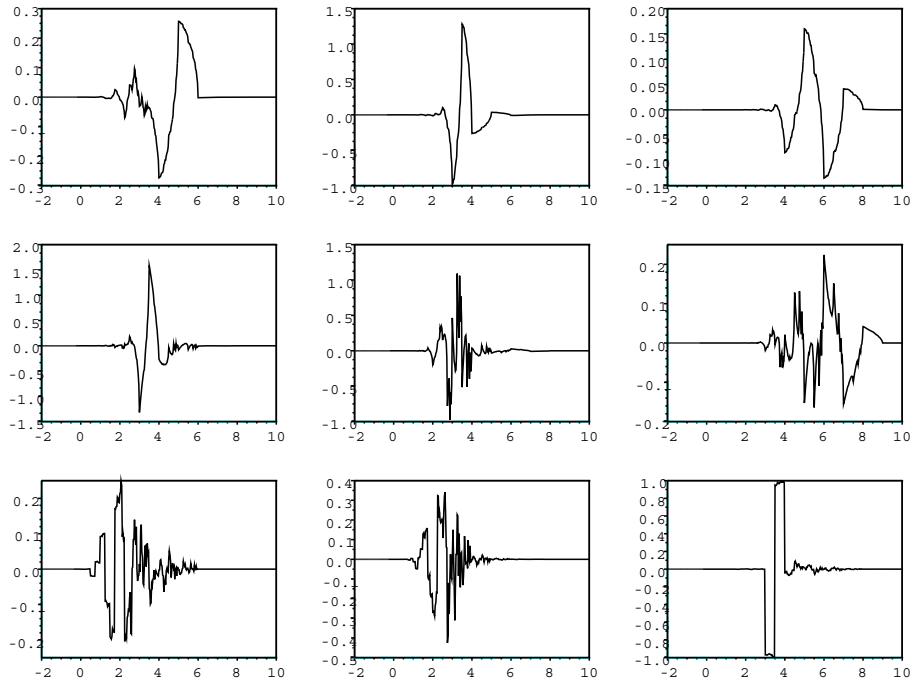


Figure 4: Orthonormal 3-channel wavelet

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